Tutorial: <https://uofglasgow.zoom.us/j/99481578600>

# Week 1

## 2C: Logic and Inequalities

Statement – a sentence which is either true or false (not both). Statements may contain variables, the value of which influences the truth of the statement.

Inclusive or: (P or Q) is true if at least one is true

Implications:

* If P, then Q:
  + Other variations: If P, then Q; Q if P; P only if Q; Q whenever P
* The statement is true when it holds, otherwise, false

Converses and Contrapositives:

* Implication is
  + P – hypothesis or assumption
  + Q – conclusion
* Converse of the implication is
  + NOT EQUIVALENT
* Contrapositive of the implication is
  + This is equivalent to the implication
* Negation is ( P and not(Q) )
  + NOT EQUIVALENT

Equivalent statements

* P and Q (two separate statements) are equivalent if they have the same truth values (visible on a truth table)

## 2C: Quantifiers

Quantifiers:

* Universal quantifier - (for all)
* Existential quantifier - (there exists)

Negating quantified statements:

* Examples:

Structure of a direct proof:

* + 1. Let a be arbitrary
  + 2. Define b by (depends on a, not c)
  + 3. Let c be arbitrary
  + 4. Now show that P(a,b,c) is true.

# Week 2

## 2C: Modulus Function, Estimation Lemma

Modulus function:

* Properties:
  + (triangle inequality)
  + if ,
  + if ,

Polynomial Estimation Lemma

* Let , and suppose with Write
* Then there exists N>0 such that
  + We may assume N is a natural number (by replacing it with nearest integer up)
  + Allows the estimation of polynomials by their “largest term”
  + If a(n) < 0, switch left and right side of lemma

# Week 3

## 5: Least Upper Bounds and Greatest Lower Bounds

Least upper bound:

* M is a least upper bound if
  + M is an upper bound
  + for all upper bounds M’,
* Lemma:
  + M is a least upper bound if and only if
  + M does not need to be in A (e.g., with an exclusive interval)

Greatest lower bound:

* m is a greatest lower bound if and only if
  + m is a lower bound
  + for all lower bounds m’,
* Lemma:
  + m is a greatest lower bound if and only if

Uniqueness

* A has at most one least upper bound and at most one greatest lower bound
* Least upper bound – supremum (sup(A))
* Greatest lower bound – infimum (inf(A))

## 6: Completeness Axiom

Completeness Axiom:

* Every non-empty subset A of which is bounded above has a least upper bound
* Remark:
  + Distinguishes between Q and
* Equivalent: Every non-empty subset A of which is bounded below has a greatest lower bound
* Proof:
  + Consider
  + B is non-empty
  + Check: B is bounded above (if A is bounded below)
  + By the completeness axiom, sup(B) exists
  + Check: m=-sup(A) is a greatest lower bound for A

Consequences:

* Archimedes’ Axiom
  + is not bounded above
    - Proof:
      * Assume M is an upper bound for
      * Then
      * If n is a natural number, then n+1 is a natural number
      * Then
      * Then
      * If M was a least upper bound, then this would be impossible. Thus, is not bounded above
* There exists
  + Idea of proof:
    - Define
    - Show that A is non-empty and bounded above, so that exists by the completeness axiom
    - Use “x is an upper bound for A” to show that
    - Use “x is the least upper bound for A” to show that

# Week 4

## 7: Sequences, Convergence

Sequence: a function N to R

* They can be bounded
* Notation:
  + Let be a real sequence
  + not a(n)

Boundedness:

* Bounded above:
* Bounded below

### Convergence of Sequences:

* For a sequence x­n and number L, xn converges to L as n tends to infinity if and only if
* Properties
  + n0 will typically depend on epsilon
  + If n0 works for epsilon, then n0 will also work for any larger value of epsilon
  + If n0 works for a specific epsilon, then any larger n0 will also work for this epsilon
  + The values of the first few terms of the sequence do not affect whether the sequence converges or its limit

## 8: Proving (Non-)Convergence by Definition

Start with , get to n

# Week 5

## 9: Properties of Limits

Uniqueness of limits

* If it is a convergent sequence, its limit is unique
* Proof:
  + Assume M, L are limits and M=/=L
  + Let (positive by assumption)
  + By definition of convergence, we find natural n1, n2 such that
  + Take n0 as the max of n1, n2
  + For
  + Contradiction

Convergent sequences are bounded

* Proof:
  + L is the limit
  + Applying definition of convergence for E=1
  + Let M < max(x1, x2, …, x(n0-1), 1+L)
  + Then xn leq M

Algebraic properties of limits

* For we have as
* as
* as
* as
* (proof by using the definition, triangle inequality)

## 10: Sandwich Principle, Standard Limits

Limits and Order Theorem:

* If x and y are sequences with limits L and M, respectively, there exists a natural N such that, for , , then
* Proof:
  + Assume L > M. Set = L – M/2
  + For we have
  + For we get
  + Contradiction to the fact that there exists N s.t.

Sandwich principle

* Sequences x, y, z. Suppose both x and z tend to L as n tends to infinity. If there exists a natural N such that
* then
* Proof:
  + Let arbitrary epsilon be positive so that
  + For
  + If

Standard limits:

* (positive alpha)
* if and only if |x|< 1
* (positive x)

# Week 6

## 11: Monotonic Sequences, Subsequences

Monotonic Sequence:

* increasing if, for all natural n,
* strictly increasing if, for all natural n,
* eventually increasing if and only if there exists natural N such that n geq N,
* eventually strictly increasing if and only if there exists a natural N such that n geq N,
* Same for opposite
* Monotonic – increasing/decreasing
  + (strictly monotonic, eventually monotonic, eventually strictly monotonic)
* Proving a sequence is (eventually) monotonic
  + Simplify and try and show that this expression is always positive/negative (for large n)
  + If x\_n > 0 for all (large) n, consider . Try and simplify this and show that it is eventually greater than or equal to 1 (or other)

Monotone Convergence Theorem:

* If a sequence is eventually increasing and bounded above or eventually decreasing and bounded below, it converges
* **Bounded monotonic sequences converge**
  + Proof:
    - If a set S is the set of all elements of an increasing sequence and it is bounded above, there exists sup(A) (completeness axiom)
    - Let positive epsilon be arbitrary. By the defining property of the supremum, there exists such that Since it’s increasing, for all n geq n\_0, we have
    - Thus, by definition of limit, as

Subsequences:

* Def: If is a sequence, a subsequence is a sequence of the form for some strictly increasing
* Limit of a sequence is L if and only if every subsequence of it converges to L
  + Proof:
    - Limit of sequence x\_n is L, x\_k\_n is a subsequence
    - arbitrary epsilon, there exists natural n\_0 such that for all n geq n\_0 |x\_n – L|< epsilon. For n geq n\_0 we have
    - (k\_n is strictly increasing k\_1 geq 1)
    - x\_k\_n – L < epsilon
    - Hence, limit of x\_k\_n is L
* Technique for proving sequences do not converge
  + Find 2 subsequences which converge to different values

## 12: Bolzano-Weierstrass Theorem, Cauchy Sequences

Term: far-seeing – a natural number n if, for all m > n,

Lemma:

* Every real sequence has a monotonic subsequence
  + Case 1: Infinitely many far-seeing numbers: strictly decreasing sequence
  + Case 2: Finitely many far-seeing numbers: increasing subsequence (by induction)

Bolzano-Weierstrass Theorem:

* Every bounded real sequence has a convergent subsequence

Cauchy sequences:

* We say a sequence is Cauchy if
* Elements of a Cauchy sequence are *eventually getting very close together*
* Properties
  + Every convergent sequence is Cauchy
    - Proof: start from def, triangle inequality
  + Every Cauchy sequence is bounded
    - Express x\_m

General Principle of Convergence:

* A real sequence is Cauchy if and only if it converges
  + Proof (Cauchy -> convergent):
    - Bolzano-Weierstrass Theorem
    - Cauchy def
    - Weird rearranging

# Week 7

## 13: Series (Convergence, Properties)

Given a sequence of real numbers a\_1, a\_2, …, the corresponding series is the sequence given by

The term s\_n is called the nth partial sum of the series. We write

for the sequence of partial sums.

Let be a series and

* The series converges if and only if the sequence of partial sums converges. In this case, we call the value of the limit the sum of the series and write for this value.
* The series diverges if and only if it does not converge
  + The “first few” terms of a series have no bearing on whether the series converges. They can, however, affect the sum.
  + A standard error is to say that converges if converges

Theorem (necessary condition for convergence):

* Let be a convergent series. Then
  + Contrapositive is true (which is an easy way to check convergence)

Proposition (example):

* The geometric series
* converges if and only if |x|< 1. When |x|< 1, the sum of the series is .

Properties of convergent series ( and )

* If for all natural n,

Proposition (example)

* The harmonic series
* diverges.

## 14: Convergence of Series (Comparison, Ratio Test)

### Comparison Test

* Let and be sequences such that there exists such that for Then
* Proof:
  + y\_n converges, then it is bounded
  + then x\_n is bounded
  + x\_n is increasing
  + then x\_n converges

For use in the comparison test:

* Geometric series: converges if and only if |x|< 1
* converges if and only if p>1
  + Proof:
    - series is leq series with p = 2
    - analyse p = 2
    - express as first term and sum of next terms
    - leq n(n+1) in denominator
    - by rearranging a lot, converges to 1
    - by comparison test, converges, as required

### Limit version of comparison test:

* Let and be sequences of eventually positive terms, and suppose Then
* Proof:
  + bounded sequence because it converges, bound is k
  + then x\_n leq ky\_n
  + if y\_n converges, x\_n converges by comparison test
  + other way round with 1/L (L=/=0)

### The Ratio Test:

* Let be a sequence.
  + Assume there exists and Then
  + Assume there exists Then
  + Proof:
    - Assume n\_0 is 1
    - Since lambda < 1, comparison with geometric series shows …, as required

### The limit version of the Ratio Test:

* Let be a sequence with Suppose that
  + If L<1, the series converges
  + If L>1, the series diverges

# Week 8

## 15: Leibniz’s Test, Absolute Convergence

Leibniz’s Test:

* Let be a sequence and suppose there exists with
  + for all (the sequence is eventually non-negative)
  + (sequence is eventually decreasing)
* Then converges
  + Proof:
    - [see lecture]

Types of convergence

* A series is absolutely convergent if and only if the series converges
  + If a series is an absolutely convergent series, it converges
    - Proof:
      * Comparison test
      * Write down
* A series is conditionally convergent if and only if converges but is not absolutely convergent

## 16: Guide for Testing Convergence, Rearrangements

Guide for testing convergence, given series :

1. Does
   1. If no, the series diverges, if yes, continue
2. Are the eventually positive?
   1. Yes
      1. Is the form of the ’s amenable to the ratio test (expressions like in the sequence)? Then consider the ratio test
      2. Otherwise, try comparison test? Useful ones: , which converges if and only if p > 1
   2. No:
      1. Do the terms alternate in sign? If so, consider Leibniz’s test
      2. Otherwise, try to see if the series is absolutely convergent. Use the above methods for

Rearranging series:

* Given a series , we can form a new series by rearrangement (must use all elements). Formally, given a bijection we consider
  + If is absolutely convergent with sum S, then all rearrangements are also absolutely convergent
  + If is conditionally convergent, then for every , there exists a rearrangement of , which converges to S. There are also rearrangements which are divergent.

# Week 9

## 17: Continuity

Definition of Continuity:

* Let Say that f is **continuous at c** if and only if
* Slogan:
  + If f is continuous, then small changes in the argument x lead to small changes in the value f(x)
* Global continuity:
  + Let f be a real function. We say that f is **continuous** if and only if f is continuous at c for all c in the domain of f. Thus, f is continuous if and only if

Framework for proving continuity at c:

1. Let epsilon be arbitrary
2. Aim: find such that
3. Simplify RHS (try to identify factors of LHS)
4. If it’s possible to write , then show that g(x) is bounded for x close to c
5. What can be said about g(x) when is small – is it bounded?

Discontinuity:

* Discontinuous at c – not continuous at c

## 18: Properties of Continuous Functions

Sequential Characterisation of Continuity

* Let f be a real function and c in its domain. Then f is continuous at c if and only if, whenever is a sequence in the domain of f with , the sequence converges to f(c)
* Slogan:
  + Continuous functions preserve the limits of convergent sequences
* Proof:
  + Definition of continuity
  + Sequence with limit c
  + Conversely:
  + Assume
  + Assume f is discontinuous at c (proof by contradiction)
  + Then
  + Set
  + Then
  + Then is a sequence s. t. but
  + Contradiction.

Properties:

Let f and g be real functions, let c be in domain f and domain of g, and suppose f and g are continuous at c. Then:

* For any , is continuous at c
  + Proof:
    - case a:
      * Constant function is continuous
    - case b:
      * Let epsilon>0
      * Continuity definition
      * As required
* is continuous at c
* is continuous at c
* If then is continuous at c

Conclusions:

* Polynomial functions and rational functions are continuous
  + Let
  + be a polynomial with real coefficients. Then the function mapping x to g(x) is continuous.
  + More generally, let
  + where g(x), h(x) are polynomials with real coefficients, and let be the set of zeros of h(x). Then the function mapping x to f(x) is continuous.
    - Proof:
      * Consider identity function (id(x)=x) – continuous function
        + Prove by definition of continuity
      * Multiplication property implies is continuous for all
      * Scalar multiple and addition properties imply that
      * is continuous.
      * Then use division property to prove the general case

Composition of Continuous Functions

* If:
  + f is continuous at c
  + g is continuous at f(c)
* Then:
  + is continuous at c
* Proof:
  + Let epsilon > 0
  + Definition of continuity for both functions, then insert

# Week 10

## Lecture 19: The Intermediate Value Theorem

The Theorem:

* Let be continuous and suppose that satisfies
* Then there exists with
* Proof:
  + Consider the case
  + Define
  + Since f(a)<d by assumption, we have , so that S is non-empty. By construction, S is bounded above by b. Therefore, c = sup(S) exists.
  + Then there exists a sequence with , s. t. as
  + Since f is continuous at c, we have as , and because ,
  + Since , we have ().
  + Now consider interval
  + Since c is the infimum of this set, we may choose in converging to c. Since is empty, , . Again, by continuity, as
  + Thus, proved.
* Applications:
  + Suppose that and or and Then there exists with

## Lecture 20: Extreme Value Theorem

Lemma:

* If is continuous, then f is bounded.
* Proof:
  + Assume f is not bounded above.
  + Then for any natural n, there exists s. t.
  + By Bolzano-Weierstrass Theorem, the sequence has a subsequence converging to some
  + Therefore, by continuity of f at c, we obtain
  + Then is bounded.
  + Suppose m is an upper bound for this sequence, then we have
  + Take such that Then
  + This is a contradiction

Extreme Value Theorem:

* If is continuous, then there exists such that, for all
* **Slogan**: Continuous functions on closed and bounded sets are **bounded** and **attain their bounds**
* Proof:
  + Consider the set
  + S is non-empty and, by previous lemma, bounded. Hence, m = sup(S) exists
  + (Contradiction) **Suppose** for all . Then we can define a function by
  + g is continuous by the algebraic properties of continuous functions
  + Hence, g is bounded by the lemma
  + So
  + Thus, is an upper bound for S, which is a contradiction of bolded statement

Theorem:

* If is continuous, then for some
* Proof:
  + By the extreme value theorem, Hence,
  + If then the set consists of a single point, then the above inclusion is an equality.
  + If : Consider the restriction of f on [u,v]: is continuous. By the intermediate value theorem, there exists such that Hence, we have
    - Therefore,
  + If : [proceed in a similar fashion]
  + Thus, proved

Continuous bijections

* A map is a bijection if it is injective and surjective
  + f is a bijection if and only if there exists a map such that
    - where and are the identity maps of X and Y, respectively
  + We write in this case and call it the inverse map
* Lemma:
  + Let be a continuous bijection. Then and or and
  + Proof:
    - According to the previous lemma, for some
    - Since f is surjective, we have , i.e. and
    - Assume and consider
      * By the intermediate value theorem, In other words, .
      * In particular, if then .
      * By the argument above, . Since this contradicts the injectivity of f. Hence,
      * If we get for some
      * Moreover, , which contradicts the injectivity of f. Therefore,
    - Assume : [do the same]
* Theorem:
  + Let be a bijective continuous map. Then
    - f and are both strictly increasing or strictly decreasing;
    - The inverse map is continuous
  + Proof:
    - Consider case
      * By the lemma above, and . Now let If then is non-empty by the intermediate theorem, which contradicts injectivity of f. Hence, , which means f is strictly increasing
    - In the other case, f is strictly decreasing
    - Inverse increasing/decreasing:
      * Assume f is strictly increasing but is not. Then with such that
      * Applying the strictly increasing function f to and gives which is a contradiction.
      * Analogous case with f being strictly decreasing
    - Inverse continuity:
      * Consider that f is strictly increasing:
        + Let be a monotonic sequence in converging to
        + Since is also strictly increasing. Then is a monotonic sequence in .
        + By the monotone convergence theorem, we have converges to some .
        + Since f is continuous, we get
        + By the uniqueness of limits, Hence, .
        + Now let be arbitrary. Consider By the argument above, as . Therefore, we obtain such that
        + Consider Then as .
        + Therefore, we obtain such that , i.e.
        + Take Then we have and i.e.
        + By the monotonicity of , we have attains its extreme values at the end points.
        + That is, for with we get .
        + In other words, is continuous at S. Since S is arbitrary, this shows that is continuous.

# Week 11

## Lecture 21: Uniform Continuity

Definition:

* A real function is called **uniformly continuous** if, for every , there exists such that, for all , we have that implies
  + Delta needs to be **independent of c**
* Any uniformly continuous function is continuous

Theorem:

* Any continuous function is uniformly continuous
  + Proof:
    - Assume f is not. Then [negation of uniform continuity].
    - In particular, taking , we find such that and The sequence is bounded, so by Bolzano-Weierstrass, it contains a convergent subsequence , with limit
    - Claim: converges to c. Proof:
      * Sandwich principle:
    - f is continuous
      * =>
      * =>
    - So there exist such that
    - So for Contradiction.